

General relativistic tidal work for Papapetrou, Weinberg and Goldberg pseudotensors

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Abstract

In 1998 Thorne claimed that all pseudotensors give the same tidal work as the Newtonian theory. In 1999, Purdue used the Landau-Lifshitz pseudotensor to calculate the tidal heating and the result matched with the Newtonian gravity. Soon after in 2001, Favata employed the same method to examine the Einstein, Bergmann-Thomson and Møller pseudotensors, all of them give the same result as Purdue did. Inspired by the work of Purdue and Favata, for the completeness, here we manipulate the tidal work for Papapetrou, Weinberg and Goldberg pseudotensors. We obtained the same tidal work as Purdue achieved. In addition, we emphasize that a suitable gravitational energy-momentum pseudotensor requires fulfill the inside matter condition and all of the classical pseudotensors pass this test except Møller. Moreover, we constructed a general pseudotensor which is modified by 13 linear artificial higher order terms combination with Einstein pseudotensor. We find that the result agrees with Thorne's prediction, i.e., relativistic tidal work is pseudotensor independent.

1 Introduction

The tidal heating or tidal work is a real physical phenomenon. Tidal heating means the net work done by an external tidal field on an isolated body [1]. A typical example is the Jupiter-Io system. The satellite Io receiving a huge gravitational energy through the tidal field from Jupiter, i.e., intensive volcanism activities are observed [2]. But how to calculate the tidal heating rate? Pseudotensor may be one of the appropriate options. In 1998 Thorne [3] claimed that all pseudotensors give the same relativistic tidal heating as the Newtonian theory. The present paper is trying to illustrate that Thorne's prediction is valid. More precisely, we have verified that the tidal work rate is indeed independent of one's choice how to localize the gravitational energy for all pseudotensors.

The tidal work rate is $\dot{W} = -\frac{1}{2}\dot{I}_{ij}E^{ij}$, where W refers to the tidal work, the dot means differentiate w.r.t. time t , I_{ij} is the mass quadrupole moment of the isolated body and E_{ij} is the tidal field of the external universe. Note that both I_{ij} and E_{ij} are time dependent, symmetric and traceless. Here we emphasize that the physical meaning of \dot{W} is the rate at which the external field does work on the isolated planet and this is an energy dissipation process which means time irreversible [4]. In contrast, there exists a recoverable process $\dot{E}_{\text{int}} \sim \frac{d}{dt}(I_{ij}E^{ij})$ which is time reversible, where E_{int} is the energy interaction between the isolated planet's quadrupolar deformation and the external tidal field. Purdue uses $E_{\text{int}} = \frac{\gamma+2}{10}I_{ij}E^{ij}$ to interpret different choices of energy localization by tuning the coefficient γ [1].

It is known that there are infinite number of pseudotensors, however, for the classical pseudotensors are several. They are Einstein [5], Landau-Lifshitz [6], Bergmann-Thomson [7], Goldberg [8], Papapetrou [9], Weinberg [10] and Møller [11]. In 1999, Purdue [1] used the Landau-Lifshitz pseudotensor to calculate the tidal heating and the result matched with the Newtonian perspective. Soon after that in 2001, Favata [12] employed the same method to examine the Einstein, Bergmann-Thomson and Møller pseudotensors, all of them give the same result as Purdue did. Inspired by the work of Purdue and Favata, for the completeness, here we manipulate the tidal

work for the Papapetrou, Weinberg and Goldberg pseudotensors, we find that all of them give the same desired tidal work as Purdue achieved.

In addition, we elucidate that a suitable gravitational energy-momentum pseudotensor requires satisfy the inside matter condition [13], i.e., see (3). It is known that all of the classical pseudotensors pass this test exclude Møller [14]. Moreover, we constructed a general pseudotensor expression which is modified by 13 linear artificial higher order terms combination with Einstein pseudotensor. We find that all the results satisfy Thorne's prediction, i.e., relativistic tidal work rate is pseudotensor independent.

2 Technical background

Here we used the same spacetime signature and the notation as in [13]: we set the geometrical units $G = c = 1$, where G and c are the Newtonian constant and speed of light. The Greek letters denote the spacetime and Latin letters refer to spatial. In principle, the classical pseudotensor [14] can be obtained from a rearrangement of the Einstein field equation: $G_{\alpha\beta} = \kappa T_{\alpha\beta}$, where constant $\kappa = 8\pi G/c^4$, $G_{\alpha\beta}$ and $T_{\alpha\beta}$ are the Einstein and stress tensors. We define the gravitational energy-momentum density pseudotensor in terms of a suitable superpotential $U_\alpha^{[\mu\nu]}$:

$$2\kappa\sqrt{-g}t_\alpha^\mu := \partial_\nu U_\alpha^{[\mu\nu]} - 2\sqrt{-g}G_\alpha^\mu. \quad (1)$$

Alternatively, one can rewrite (1) as $\partial_\nu U_\alpha^{[\mu\nu]} = 2\sqrt{-g}(G_\alpha^\mu + \kappa t_\alpha^\mu)$. Using the Einstein equation, the total energy-momentum density complex can be defined as

$$\mathcal{T}_\alpha^\mu := \sqrt{-g}(T_\alpha^\mu + t_\alpha^\mu) = (2\kappa)^{-1}\partial_\nu U_\alpha^{[\mu\nu]}. \quad (2)$$

This total energy-momentum density is automatically conserved as $\partial_\mu \mathcal{T}_\alpha^\mu \equiv 0$ which can be classified into two parts

$$\partial_\nu U_\alpha^{[\mu\nu]} = 2\sqrt{-g}G_\alpha^\mu, \quad \partial_\nu U_\alpha^{[\mu\nu]} = 2\sqrt{-g}\kappa t_\alpha^\mu. \quad (3)$$

The first part indicates inside matter and the second piece belongs to vacuum gravity [13]. For the condition of interior mass-energy density, it is known all the classical pseudotensors give the standard result $2G_\alpha^\mu$, but only Møller failed [14], i.e., $\partial_\nu U_\alpha^{\mu\nu} = R_\alpha^\mu$. More precisely, though the Møller pseudotensor gives the desired tidal work, it is disqualified as a satisfactory description of energy-momentum. Although there are infinite numbers of pseudotensors, we can remove some forbidden freedom [15, 16]. Thus, perhaps, the Freud superpotential [5] is a simpler expression to demonstrate the tidal work:

$${}_F U_\alpha^{[\mu\nu]} = \sqrt{-g}[(\delta_\alpha^\mu \Gamma^{\lambda\nu}{}_\lambda + \delta_\alpha^\nu \Gamma^{\mu\lambda}{}_\lambda + \Gamma^{\nu\mu}{}_\alpha) - (\mu \leftrightarrow \nu)]. \quad (4)$$

There are three kinds of superpotential: $U_\alpha^{[\mu\nu]}$, $U^{\alpha[\mu\nu]}$ and $H^{[\alpha\beta][\mu\nu]}$. The total energy-momentum complex can be obtained as follows

$$2\kappa\mathcal{T}_\alpha^\mu = \partial_\nu U_\alpha^{[\mu\nu]}, \quad 2\kappa\mathcal{T}^{\alpha\mu} = \partial_\nu U^{\alpha[\mu\nu]}, \quad 2\kappa\mathcal{T}^{\alpha\mu} = \partial_{\beta\nu}^2 H^{[\alpha\beta][\mu\nu]}. \quad (5)$$

In vacuum, t_0^j is the gravitational energy flux and the tidal work can be computed in either ways

$$\frac{dW}{dt} = \frac{1}{2\kappa} \oint_{\partial V} \sqrt{-g} t_0^j \hat{n}_j r^2 d\Omega, \quad \frac{dW}{dt} = -\frac{1}{2\kappa} \oint_{\partial V} \sqrt{-g} t^{0j} \hat{n}_j r^2 d\Omega, \quad (6)$$

where $r \equiv \sqrt{\delta_{ab}x^a x^b}$ is the distance from the body in its local asymptotic rest frame and $\hat{n}_j \equiv x_j/r$ is the unit radial vector. In our tidal heating calculation, we will use the

deDonder (harmonic) gauge [1], $\partial_\beta(\sqrt{-g}g^{\alpha\beta}) = 0$, which is equivalently represented as $\Gamma^{\alpha\beta}_\beta = 0$. From this, we deduce the following identity when $\alpha = 0$:

$$\frac{\partial h_{00}}{\partial x^0} = \frac{1}{2}\eta^{cd}\frac{\partial h_{0c}}{\partial x^d}. \quad (7)$$

The metric tensor can be decomposed as $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, and its inverse $g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$. Then we have the following physical expressions [1]:

$$h_{00} = \frac{2M}{r} + \frac{3}{r^5}I_{ij}x^i x^j - E_{ij}x^i x^j, \quad h_{0j} = \frac{2}{r^3}\dot{I}_{ij}x^i - \frac{10}{21}\dot{E}_{ik}x^i x^k x_j + \frac{4}{21}\dot{E}_{ij}x^i r^2, \quad (8)$$

note that $h_{ij} = \delta_{ij}h_{00}$. Here we remark that (7) can be obtained directly from differentiation using (8).

3 Papapetrou, Weinberg, Goldberg and general expression pseudotensors

Here we consider the Papapetrou, Weinberg and Goldberg pseudotensors, we also constructed a general pseudotensor expression. We find that all of them give the same desired tidal work rate.

3.1 Papapetrou pseudotensor

The Papapetrou superpotential is defined as $H_P^{[\mu\nu][\alpha\beta]} := -\sqrt{-g}g^{\rho\pi}\eta^{\tau\gamma}\delta_{\pi\gamma}^{\nu\mu}\delta_{\rho\tau}^{\alpha\beta}$, equivalently one can use $U_P^{\alpha[\mu\nu]} \equiv \partial_\beta H^{[\mu\nu][\alpha\beta]}$. In terms of the Bergmann-Thomson superpotential $U_{BT}^{\alpha[\mu\nu]}$, we have

$$U_P^{\alpha[\mu\nu]} := U_{BT}^{\alpha[\mu\nu]} - \sqrt{-g}(g^{\rho\tau}h^{\pi\gamma}\Gamma^\sigma_{\lambda\pi} + g^{\rho\pi}h^{\sigma\tau}\Gamma^\gamma_{\lambda\pi})\delta_{\tau\gamma}^{\mu\nu}\delta_{\rho\sigma}^{\lambda\alpha}, \quad (9)$$

where $U_{BT}^{\alpha[\mu\nu]} := -\sqrt{-g}g^{\alpha\beta}g^{\pi\sigma}\Gamma^\tau_{\lambda\pi}\delta_{\tau\sigma\beta}^{\lambda\mu\nu}$. The Papapetrou pseudotensor becomes

$$\begin{aligned} t_P^{\alpha\mu} &= t_{BT}^{\alpha\mu} + (g^{\alpha\mu}\Gamma^\lambda_{\lambda\beta} - 2\Gamma^{\alpha\mu}_\beta - 2\Gamma^{\mu\alpha}_\beta)\Gamma^{\beta\nu}_\nu + (\Gamma^{\alpha\nu}_\nu + \Gamma^{\nu\alpha}_\nu)\Gamma^{\mu\pi}_\pi \\ &\quad + (g^{\alpha\mu}\Gamma^\lambda_{\lambda\beta} - \Gamma^{\alpha\mu}_\beta - \Gamma^{\beta\alpha}_\mu - 2\Gamma^{\mu\alpha}_\beta)\Gamma^{\nu\beta}_\nu + (\Gamma^{\alpha\beta\nu} + \Gamma^{\beta\nu\alpha})\Gamma^\mu_{\beta\nu} + (\Gamma^{\alpha\beta\nu} + \Gamma^{\nu\beta\alpha})\Gamma^\mu_{\beta\nu} \\ &\quad + (g^{\alpha\mu}h^{\lambda\pi} - h^{\alpha\mu}g^{\lambda\pi})\Gamma^\nu_{\lambda\pi,\nu} - (g^{\pi\mu}h^{\lambda\nu} - h^{\pi\mu}g^{\lambda\nu})\Gamma^\alpha_{\lambda\pi,\nu} - (g^{\alpha\pi}h^{\lambda\nu} - h^{\alpha\pi}g^{\lambda\nu})\Gamma^\mu_{\lambda\pi,\nu} \\ &= -(\Gamma^{\alpha\mu}_\beta + \Gamma^{\mu\alpha}_\beta)\Gamma^{\beta\nu}_\nu + \Gamma^{\alpha\nu}_\nu\Gamma^{\pi\mu}_\pi + \Gamma^{\nu\alpha}_\nu\Gamma^{\mu\pi}_\pi + g^{\alpha\mu}(\Gamma^{\nu\pi}_\rho\Gamma^\rho_{\pi\nu} + \Gamma^\lambda_{\lambda\beta}\Gamma^{\nu\beta}_\nu) \\ &\quad - 2(\Gamma^{\alpha\mu}_\beta + \Gamma^{\mu\alpha}_\beta)\Gamma^{\nu\beta}_\nu + 2\Gamma^{\alpha\beta\nu}\Gamma^\mu_{\beta\nu} + (g^{\alpha\mu}h^{\lambda\pi} - h^{\alpha\mu}g^{\lambda\pi})\Gamma^\nu_{\lambda\pi,\nu} \\ &\quad - (g^{\pi\mu}h^{\lambda\nu} - h^{\pi\mu}g^{\lambda\nu})\Gamma^\alpha_{\lambda\pi,\nu} - (g^{\alpha\pi}h^{\lambda\nu} - h^{\alpha\pi}g^{\lambda\nu})\Gamma^\mu_{\lambda\pi,\nu}, \end{aligned} \quad (10)$$

where the Bergmann-Thomson pseudotensor $t_{BT}^{\alpha\mu}$ expression can be found in [12]. Alternatively, we use our own representation

$$\begin{aligned} t_{BT}^{\alpha\mu} &= -(g^{\alpha\mu}\Gamma^\pi_{\pi\rho} - \Gamma^{\mu\alpha}_\rho - \Gamma^{\alpha\mu}_\rho)\Gamma^{\rho\nu}_\nu + \Gamma^{\alpha\nu}_\nu(\Gamma^{\pi\mu}_\pi - \Gamma^{\mu\pi}_\pi) + g^{\alpha\mu}\Gamma^{\nu\rho\pi}\Gamma_{\rho\pi\nu} \\ &\quad + (\Gamma^{\rho\alpha\mu} - \Gamma^{\alpha\mu\rho})\Gamma^\pi_{\pi\rho} + \Gamma^{\alpha\pi\rho}(\Gamma^\mu_{\pi\rho} - \Gamma_{\pi\rho}^\mu) - \Gamma^{\rho\pi\alpha}(\Gamma^\mu_{\rho\pi} + \Gamma_{\pi\rho}^\mu). \end{aligned} \quad (11)$$

Equation (10) indicates the gravitational energy-momentum, for inside matter we have $t_P^{\alpha\mu} = 2G^{\alpha\mu}$. The Papapetrou pseudotensor is, in some sense, a modification of Bergmann-Thomson pseudotensor by introducing a flat metric $\eta_{\alpha\beta}$. The corresponding tidal work for Papapetrou pseudotensor is

$$\frac{dW_P}{dt} = -\frac{1}{10}\frac{d}{dt}(I_{ij}E^{ij}) - \frac{1}{2}\dot{I}_{ij}E^{ij}. \quad (12)$$

Here we have the desired result for the tidal work $-\frac{1}{2}\dot{I}_{ij}E^{ij}$ and the interaction energy $E_{\text{int}} = \frac{\gamma+2}{10}I_{ij}E^{ij}$ requires $\gamma = -3$. Even though the Papapetrou and Landau-Lifshitz pseudotensors have different expressions, they have the same choice for the energy localization. Moreover, like the Landau-Lifshitz pseudotensor, the Papapetrou pseudotensor also a symmetric pseudotensor. This implies that the construction of a conserved total energy-momentum complex is allowed [12].

3.2 Weinberg pseudotensor

The superpotential for Weinberg(W) is $H_W^{[\mu\nu][\alpha\beta]} := -\sqrt{-\eta}(\eta^{\xi\rho}\eta^{\kappa\tau} - \frac{1}{2}\eta^{\xi\kappa}\eta^{\rho\tau})g_{\rho\tau}\eta^{\lambda\sigma}\delta_{\xi\lambda}^{\mu\nu}\delta_{\kappa\sigma}^{\alpha\beta}$. Alternative representation for this superpotential is

$$U_W^{\alpha[\mu\nu]} := U_{W_0}^{\alpha[\mu\nu]} + \left\{ \sqrt{-\eta} \left[\begin{aligned} &(g^{\alpha\mu}h^{\nu\beta} + h^{\alpha\mu}g^{\nu\beta})\Gamma_{\beta\lambda}^{\lambda} + g^{\alpha\mu}h^{\rho\tau}\Gamma_{\rho\tau}^{\nu} + g^{\alpha\nu}h^{\mu\rho}\Gamma_{\rho}^{\beta}{}_{\beta} \\ &+(g^{\alpha\nu}h^{\beta\lambda} + h^{\alpha\nu}g^{\beta\lambda} - g^{\alpha\lambda}h^{\nu\beta} - h^{\alpha\lambda}g^{\nu\beta})\Gamma_{\beta\lambda}^{\mu} + h^{\nu\rho}\Gamma_{\rho}^{\mu}{}_{\alpha\mu} \end{aligned} \right] - (\mu \leftrightarrow \nu) \right\}, \quad (13)$$

where $U_{W_0}^{\alpha[\mu\nu]} := -\sqrt{-\eta}g^{\alpha\beta}g^{\pi\sigma}\Gamma_{\lambda\pi}\delta_{\tau\sigma\beta}^{\lambda\mu\nu}$. The Weinberg pseudotensor is

$$t_W^{\alpha\mu} = (2\Gamma^{\beta\alpha\mu} - g^{\alpha\mu}\Gamma^{\beta\pi}{}_{\pi})\Gamma_{\beta}^{\nu}{}_{\nu} + g^{\alpha\mu}\Gamma^{\rho\beta\nu}\Gamma_{\rho\beta\nu} - 2\Gamma^{\beta\nu\alpha}\Gamma_{\beta\nu}^{\mu} + 2h^{\beta\nu}R^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu}, \quad (14)$$

which is symmetric in α and μ . Inside matter we have $t_W^{\alpha\mu} = 2G^{\alpha\mu}$. In vacuum we have the tidal work rate

$$\frac{dW_W}{dt} = \frac{1}{10}\frac{d}{dt}(I_{ij}E^{ij}) - \frac{1}{2}\dot{I}_{ij}E^{ij}. \quad (15)$$

Again we have the desired tidal work $-\frac{1}{2}\dot{I}_{ij}E^{ij}$ and the interaction energy $E_{\text{int}} = \frac{\gamma+2}{10}I_{ij}E^{ij}$ indicates $\gamma = -1$.

3.3 Goldberg pseudotensor

Goldberg had changed the weighting factor $\sqrt{-g}$ to be an arbitrary class density for Einstein and Landau-Lifshitz pseudotensors [8, 14]. We discovered that the tidal work remains unchange if changing the weighting factor density $(\sqrt{-g})^n$, where n is real and finite. However, it does change the sign of internal energy and interaction energy. The detail calculation are follows: the first case of the Goldberg superpotential:

$$_G U_{\alpha}^{[\mu\nu]} := -\sqrt{-g}^n g^{\beta\sigma}\Gamma^{\tau}{}_{\lambda\beta}\delta_{\tau\sigma\alpha}^{\lambda\mu\nu}, \quad (16)$$

The corresponding gravitational energy-momentum density is

$$_G t_{\alpha}^{\mu} = \sqrt{-g}^n \left[\begin{aligned} &n(\Gamma^{\beta}{}_{\beta\alpha}\Gamma^{\mu\lambda}{}_{\lambda} - \delta_{\alpha}^{\mu}\Gamma^{\beta}{}_{\beta\nu}\Gamma^{\nu\lambda}{}_{\lambda} + \Gamma^{\nu\mu}{}_{\alpha}\Gamma^{\beta}{}_{\beta\nu} - \Gamma^{\beta}{}_{\beta\alpha}\Gamma^{\lambda\mu}{}_{\lambda}) - 2\Gamma^{\beta\nu}{}_{\alpha}\Gamma^{\mu}{}_{\beta\nu} \\ &+ \delta_{\alpha}^{\mu}\Gamma^{\beta\lambda}{}_{\nu}\Gamma^{\nu}{}_{\beta\lambda} + (n-1)\delta_{\alpha}^{\mu}\Gamma^{\nu}{}_{\nu\beta}\Gamma^{\lambda\beta}{}_{\lambda} - (n-2)\Gamma^{\mu}{}_{\alpha\beta}\Gamma^{\nu\beta}{}_{\nu} \end{aligned} \right] \quad (17)$$

Checking the value of inside matter, we have the standard result $_G t_{\alpha}^{\mu} = 2G_{\alpha}^{\mu}$. The corresponding tidal work is

$$\frac{dW_G}{dt} = \frac{5-2n}{10}\frac{d}{dt}(I_{ij}E^{ij}) - \frac{1}{2}\dot{I}_{ij}E^{ij}. \quad (18)$$

Clearly, equation (18) recovers the result for the Einstein pseudotensor when $n = 1$.

Similarly, the second case of the Goldberg superpotential is

$$U_G^{\alpha[\mu\nu]} := -(-g)^n g^{\alpha\beta}g^{\pi\sigma}\Gamma_{\lambda\pi}\delta_{\tau\sigma\beta}^{\lambda\mu\nu}. \quad (19)$$

The corresponding gravitational energy-momentum density is

$$\begin{aligned}
& t_G^{\alpha\mu} \\
& = (-g)^n \left\{ (\Gamma^{\alpha\mu}_{\beta} + \Gamma^{\mu\alpha}_{\beta} - 2ng^{\alpha\mu}\Gamma^{\gamma}_{\gamma\beta})\Gamma^{\beta\nu}_{\nu} + [(2n-1)\Gamma^{\nu\alpha}_{\nu} - \Gamma^{\alpha\nu}_{\nu}]\Gamma^{\mu\pi}_{\pi} + \Gamma^{\alpha\nu}_{\nu}\Gamma^{\pi\mu}_{\pi} \right\} \\
& + (-g)^n \left\{ [(2n-1)(g^{\alpha\mu}\Gamma^{\gamma\nu}_{\gamma} - \Gamma^{\mu\alpha\nu}) + 2n\Gamma^{\nu\alpha\mu} - \Gamma^{\alpha\mu\nu}]\Gamma^{\pi}_{\nu\pi} + g^{\alpha\mu}\Gamma^{\beta\nu}_{\rho}\Gamma^{\rho}_{\beta\nu} \right. \\
& \left. + (1-2n)\Gamma^{\nu\alpha}_{\nu}\Gamma^{\pi\mu}_{\pi} + (\Gamma^{\alpha\beta}_{\nu} - \Gamma^{\beta\alpha}_{\nu})\Gamma^{\mu\nu}_{\beta} - (\Gamma^{\alpha\nu}_{\beta} + \Gamma^{\nu\alpha}_{\beta})\Gamma^{\beta\mu}_{\nu} \right\}. \quad (20)
\end{aligned}$$

Inside matter we have $t_G^{\alpha\mu} = 2G^{\alpha\mu}$ and the tidal work in vacuum is

$$\frac{dW_G}{dt} = \frac{3-4n}{10} \frac{d}{dt} (I_{ij}E^{ij}) - \frac{1}{2} \dot{I}_{ij}E^{ij}. \quad (21)$$

Once again, we recover the result for the Landau-Lifshitz pseudotensor when $n = 1$.

3.4 General pseudotensor expression

Thorne had laid down a statement that all pseudotensors give the same tidal work rate [3]. This means that the tidal work expression is pseudotensor independent of one's choice how to localize the gravitational energy. As far as the classical pseudotensors are concerned, after the examinations by Purdue [1], Favata [12] and the present paper, we clarified that Thorne's prediction is correct. Here we demonstrate that it is still true even for a general form of pseudotensor. For simplicity, we use Freud superpotential as a leading term and consider the 13 linear artificial higher order terms combination as a modification:

$$U_{\alpha}^{[\mu\nu]} := {}_F U_{\alpha}^{[\mu\nu]} + \sqrt{-g} \sum_{i=1}^{13} k_i U_i, \quad (22)$$

where k_i are finite constants and the explicit extra terms are

$$\begin{aligned}
U_1 &= \delta_{\alpha}^{\mu} h^{\beta\nu} \Gamma^{\lambda}_{\lambda\beta}, & U_2 &= h^{\mu\pi} \Gamma_{\alpha\pi}^{\nu}, & U_3 &= \delta_{\alpha}^{\mu} h^{\beta\lambda} \Gamma^{\nu}_{\beta\lambda}, & U_4 &= h_{\alpha}^{\mu} \Gamma^{\lambda\nu}_{\lambda}, \\
U_5 &= h_{\alpha\lambda} \Gamma^{\nu\mu\lambda}, & U_6 &= \delta_{\alpha}^{\mu} h \Gamma^{\lambda\nu}_{\lambda}, & U_7 &= \delta_{\alpha}^{\mu} h^{\beta\lambda} \Gamma_{\beta\lambda}^{\nu}, & U_8 &= h \Gamma^{\nu\mu}_{\alpha}, \\
U_9 &= \delta_{\alpha}^{\mu} h \Gamma^{\nu\lambda}_{\lambda}, & U_{10} &= \delta_{\alpha}^{\mu} h^{\beta\nu} \Gamma_{\beta}^{\lambda\lambda}, & U_{11} &= h_{\alpha}^{\mu} \Gamma^{\nu\lambda}_{\lambda}, & U_{12} &= h^{\mu\pi} \Gamma^{\nu}_{\alpha\pi}, \\
U_{13} &= h^{\mu\pi} \Gamma_{\pi\alpha}^{\nu}.
\end{aligned} \quad (23)$$

where μ and ν are anti-symmetric such that every U_i is subtracted by an extra term follows the swaped μ and ν indices. In (22), consider inside matter, the leading term Freud superpotential gives the standard value $2G_{\alpha}^{\mu}$, but all U_i contribute null result. However, we find U_1 to U_8 alter the values of the internal energy and interaction energy, but U_9 to U_{13} give vanishing values. In other words, U_1 to U_8 affect the choice how to localize the gravitational energy, however U_9 to U_{13} do not. One minor thing might need to clarify that if we change the density of the weighting factor $\sqrt{-g}$ in (22), it does not affect the result of the tidal work as mentioned in section 3.3. After a straightforward manipulation, the tidal work is

$$\frac{dW}{dt} = \frac{3-\alpha}{10} \frac{d}{dt} (I_{ij}E^{ij}) - \frac{1}{2} \dot{I}_{ij}E^{ij}, \quad (24)$$

provided $\alpha = k_1 + k_2 - k_3 - k_4 - k_5 + 2k_6 + 2k_7 + 2k_8$. Thus, equation (24) illustrates that the gravitational energy localization is E_{int} dependence, while the tidal work rate is unique. Thorne's prediction is correct.

4 Conclusion

Thorne claimed that all pseudotensors give the same tidal work as the Newtonian theory. Purdue used the Landau-Lifshitz classical pseudotensor to calculate the tidal work and the result matched with the Newtonian perspective. Later Favata employed the same method to examine the Einstein, Bergmann-Thomson and Møller classical pseudotensors, all of them give the same result as Purdue did. Inspired by the work of Purdue and Favata, for the completeness, we take into account the rest of the three classical pseudotensors: Papapetrou, Weinberg and Goldberg. We find that these three pseudotensors give the same desired result as Purdue obtained. In addition, we emphasize that a suitable gravitational energy-momentum pseudotensor requires satisfy the inside matter condition and all of them pass this test but only Møller failed. Moreover, we constructed a general pseudotensor expression which is modified by 13 linear artificial higher order terms combination with the Einstein pseudotensor. This general pseudotensor fulfills the conditions of inside matter. Consequently, we have verified that Thorne's assertion is valid, i.e., tidal work is independent of one's choice how to localize the gravitational energy for all kinds of pseudotensor.

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